International Workshop on Stochastic Weather Generators

Reduced flow models from a stochastic Navier-Stokes theorem

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Plan

PLAN

INTRODUCTION
- Why using stochastic fluid dynamic ?
- Why a reduced model ?

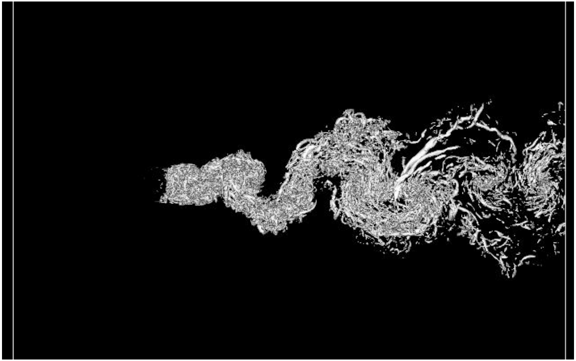
STOCHASTIC FLUID DYNAMIC MODEL

REDUCED MODEL

RESULTS

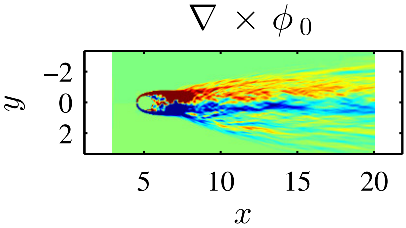
CONCLUSION

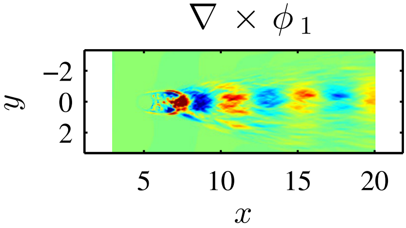
WHY USING STOCHASTIC FLUID DYNAMIC ?



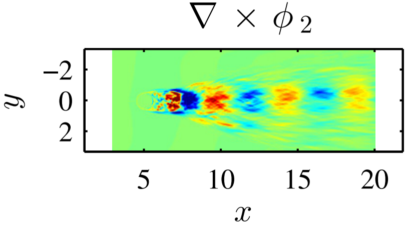
- Usual advantages of stochastic methods
- Fluid dynamics, and especially geophysical fluid dynamics, is too complex to be solved at all scales
- Chaotic system
- Non-unique solution

Why a reduced model ?





- Simplified simulations
- Decomposition of the physic



PLAN

CLASSICAL LARGE EDDY APPROACH

Numericians approach :

$$\frac{\partial \overline{T}}{\partial t} + \nabla \cdot \left(\overline{T} \overline{u} \right) = \nabla \cdot \left(-\overline{u' T'} \right)$$

Classical Large Eddy approach

Numericians approach :

$$\frac{\partial \overline{T}}{\partial t} + \nabla \cdot \left(\overline{T} \bar{u} \right) = \nabla \cdot \left(-\overline{u' T'} \right) = \frac{\sigma^2}{2} \triangle \overline{T}$$

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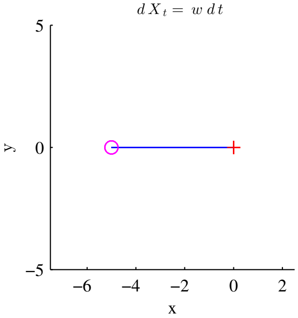
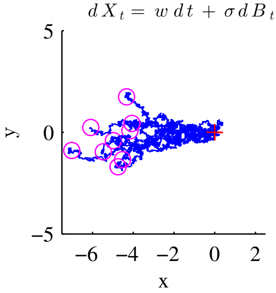




STOCHASTIC REPRESENTATION OF THE REYNOLDS-TRANSPORT THEOREM

Notations :

- $dX_t = w(X_t, t)dt + \sigma(X_t, t)dB_t$
- B_t an $\mathbb{I}_d$-cylindrical Wiener process (Da Prato & Zabczyk (1992))
- $\sigma(., t)$ an Hilbert-Schmidt operator on $(L^2(\mathbb{R}^d))^d$ defined by its kernel $\breve{\sigma}(., ., t)$

Thus,

$$\sigma(x, t)dB_t \sim \mathbf{GP}(0, \delta(t - t')\sigma(x, t)\sigma^t(x', t)dt)$$

- Local covariance matrix
  $a(x, t) \stackrel{\triangle}{=} \sigma(x, t)\sigma^t(x, t) = \int_\Omega \breve{\sigma}(x, z, t)\breve{\sigma}^t(x, z, t)dz$

Stochastic representation of the Reynolds-transport theorem

Theorem

$V(t)$ being a material element,

$$d_t \int_{V(t)} q dx = \int_{V(t)} \left(d_t q + \nabla \cdot \left(q dX_t + q\sigma(\nabla \cdot \sigma)^t dt - \frac{1}{2} \nabla \cdot (aq)^t dt \right) \right) dx$$

Proof : Mémin (2014)

Mass continuity

$$
\begin{aligned}
d_t \int_{V(t)} \rho dx &= d_t m(t) \\
&= 0
\end{aligned}
$$

Thus,

Theorem

$$0 = d_t \rho + \nabla \cdot \left(\rho dX_t + \rho \sigma (\nabla \cdot \sigma)^t dt - \frac{1}{2} \nabla \cdot (a\rho)^t dt \right)$$

Application : Mean field

$$\frac{\partial \mathbb{E}(\rho)}{\partial t} + \nabla \cdot (\mathbb{E}(\rho) w^*) = \nabla \cdot \left(\frac{1}{2} a \nabla \mathbb{E}(\rho) \right)$$

with

$$w^* = w + \sigma (\nabla \cdot \sigma)^t - \frac{1}{2} (\nabla \cdot a)^t$$

Stochastic Navier-Stokes model

Theorem

If w has finite variations and $\int_0^t p dt'$ is replaced by $\int_0^t p' dt' + d\hat{p}$ then

$$\rho\left(\frac{\partial w}{\partial t} + (w\cdot\nabla)w\right) = \boldsymbol{\tau}(\boldsymbol{w}) + \rho g - \nabla p' + f_V(w)$$

$$\rho(\sigma dBt\cdot\nabla)w = -\nabla d\hat{p} + f_V(\sigma)dBt$$

where

$$\begin{cases} \forall k, \tau_k(w) &= \frac{1}{2}\left(\nabla\cdot(\nabla\cdot(\rho a w_k))^t - \nabla\cdot(\nabla\cdot(\rho a))^t w_k\right) \\ & \quad -\rho\left((\nabla\cdot\sigma)\sigma^t\nabla\right) w_k \\ f_V(g) &= \mu\left(\nabla^2 g + \frac{1}{3}\nabla(\nabla\cdot g)\right) \end{cases}$$

Proof : Mémin (2014)

STOCHASTIC NAVIER-STOKES MODEL

Applications :

- Large eddies simulation
- Uncertainty quantification
- Filtering
- Mixing diagnostics

...

PLAN

Principle

- Using both a model and data (N snapshots).
- Galerkin projection :

$$u(.,t) \in \mathbf{Span}(\phi_0, ..., \phi_n)$$

- Coefficients of the decomposition, b_i, are time-dependent.
- $\frac{\partial u}{\partial t} = I + L(u) + C(u,u)$ (a PDE) becomes :

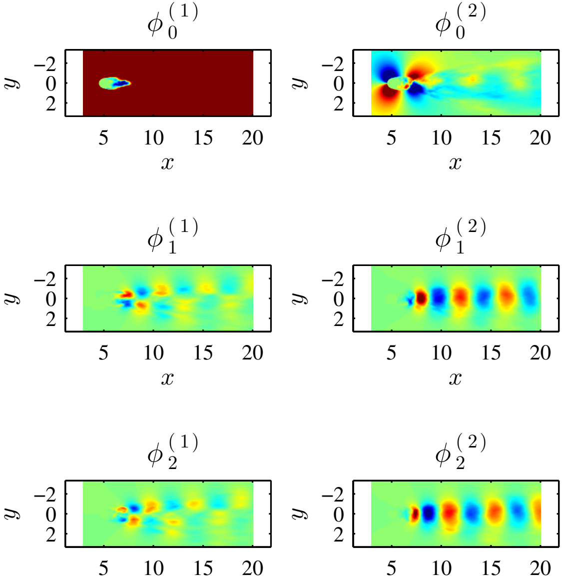
$$\forall i, \frac{db_i}{dt} = \left(\int_\Omega \phi_i \cdot I \right) + \sum_{p=0}^{n} \left(\int_\Omega \phi_i \cdot L(\phi_p) \right) b_p + \sum_{p,q=0}^{n} \left(\int_\Omega \phi_i \cdot C(\phi_p, \phi_q) \right) b_p b_q \qquad \text{(ODEs)}$$

PROPER ORTHOGONAL DECOMPOSITION

- Analogue to PCA apply to $(u(.,t_1),...,u(.,t_N))$:
  Analogue of variables : Values on points of the space
  Analogue of realizations : Snapshots
- Modes $\phi_i(x)$ are sorted regarding an energetic criterion

$$u'(.,t) \stackrel{\triangle}{=} u(.,t) - \underbrace{\bar{u}}_{\stackrel{\triangle}{=}\phi_0} \approx \sum_{i=1}^{N} b_i(t)\phi_i \approx \sum_{i=1}^{n} b_i(t)\phi_i$$

Proper Orthogonal Decomposition



CLASSICAL APPROACHES

$$\forall i, \frac{db_i}{dt} = i_i + l_{\cdot i} b_{1:n} + b_{1:n}^t c_{\cdot \cdot i} b_{1:n}$$

Problem :
Non-linear systems require too many modes.

Classical solutions : Eddy viscosity

$$l_{\cdot i} \leftarrow \frac{\nu}{\nu_m} l_{\cdot i}$$

- Constant ν (Aubry *et al.* (1988))
- Modal ν_i (Rempfer & Fasel (1994))
- ν or $\nu_i \propto \|b_{1:n}\|_2$ (Östh *et al.* (2014))
- $\nu \propto f\left(\|b_{1:n}\|_2^2\right)$ (Protas *et al.* (2014))

OUR APPROACH

Our approach :

- $dX_t = w dt + \sigma dB_t$
- $w = \sum_{i=0}^{n} b_i \phi_i$
  (in the truncated subspace)
- $\sum_{i=n+1}^{N} b_i \phi_i dt$ a realization of σdB_t
  (in the complementary "small-scale" subspace)

Estimation of a

We look for a decomposition of $a = \sum_{i=0}^{n} b_i(t) z_i(x)$.

- Reduced subspace with coherent time scales
- No need to reconstruct a
- Autonomous system

Indeed, $\frac{\partial w}{\partial t} = I + L(w) + C(w,w) + D(a,w)$ becomes :

$$\frac{db_i}{dt} = \int_\Omega \phi_i \cdot \left(I + \sum_{p=0}^{n} L(\phi_p) b_p + \sum_{p,q=0}^{n} \left(C(\phi_p, \phi_q) + D(\phi_p, z_q) \right) b_p b_q \right)$$

ESTIMATION OF a

$$\forall x \in \mathbb{R}^d, \ \sum_{i=n+1}^{N} b_i(t)\phi_i(x)\Delta t = \left(u(x,t) - w(x,t)\right)\Delta t$$

realization of $d\tilde{X}_t^x \overset{\triangle}{=} \sigma(x,t)dB_t$ (reminder : $a = \sigma\sigma^t$)

$$\begin{aligned} z_i(x) &= \int_0^T \frac{b_i(t)}{\lambda_i T} a(x,t)dt \\ &= \int_0^T \frac{b_i(t)}{\lambda_i T} d < \tilde{X}^x, \left(\tilde{X}^x\right)^t >_t \\ &= \mathbb{P} - \lim_{\Delta t \to 0} \sum_{t_i=0}^{T} \frac{b_i(t_i)}{\lambda_i T} \left(\tilde{X}^x_{t_{i+1}} - \tilde{X}^x_{t_i}\right)\left(\tilde{X}^x_{t_{i+1}} - \tilde{X}^x_{t_i}\right)^t \end{aligned}$$

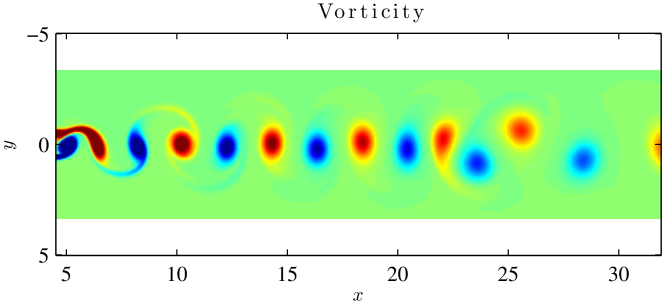
- Estimator with good statistical properties (Rao (1999)) (when the projection subset is well chosen)

Plan

Data

Wake behind a cylinder

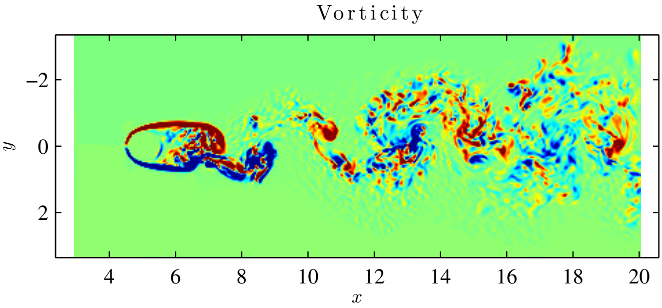
- at Reynolds 300 : 2D flow



Data

Wake behind a cylinder

- at Reynolds 300 : 2D flow
- at Reynolds 3900 : 3D flow



Turbulence modes z_i

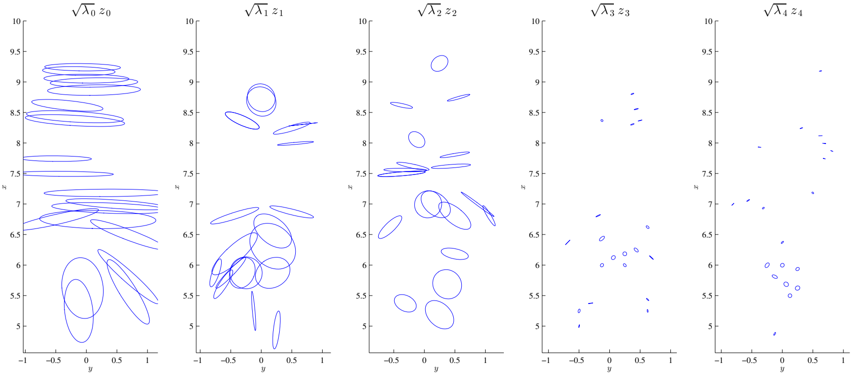


Figure : Local spectral representations of the matrix $z_i\,(x, y, 0)$, normalized by $\sqrt{\lambda_i} = \sqrt{\overline{b_i^2}}$, from the 2D flow (Re=300).

Turbulence modes z_i

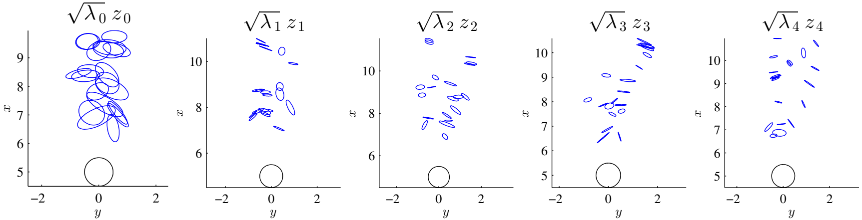


Figure : Local spectral representations of the matrix $z_i\,(x, y, 0)$, normalized by $\sqrt{\lambda_i} = \sqrt{\overline{b_i^2}}$, for some some point $(x, y, 0)$ of the horizontal section at $z = 0$, from the 3D flow (Re=3900).

RECONSTRUCTIONS OF TEMPORAL MODES

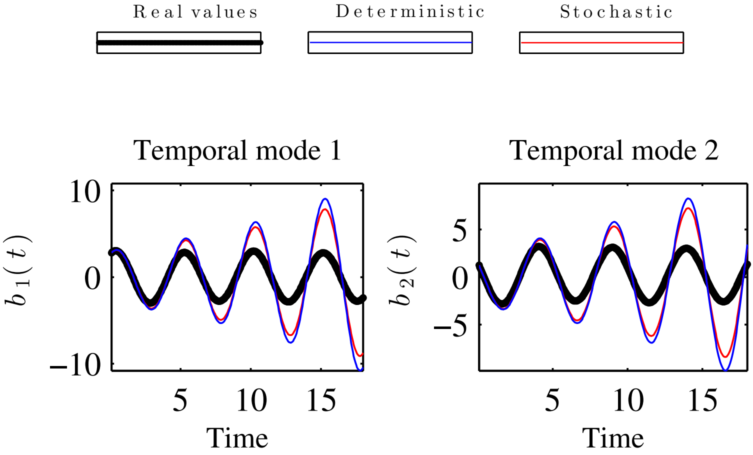


FIGURE : n=2 and Re=3900 (3D flow).

RECONSTRUCTIONS OF TEMPORAL MODES

Increase of a estimation :

$\frac{\partial w}{\partial t} = I + L(w) + C(w,w) + D(a,w)$ becomes :

$$\frac{db_i}{dt} = \int_\Omega \phi_i \cdot \left(I + \sum_{p=0}^{n} L(\phi_p) b_p + \sum_{p,q=0}^{n} \big(C(\phi_p,\phi_q) + \alpha\, D(\phi_p, z_q) \big) b_p b_q \right)$$

RECONSTRUCTIONS OF TEMPORAL MODES

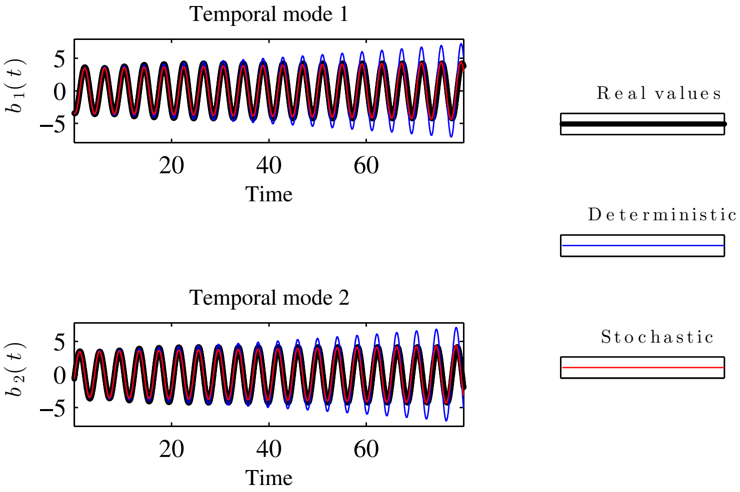


FIGURE : n=2 and Re=300 (2D flow).

RECONSTRUCTIONS OF TEMPORAL MODES

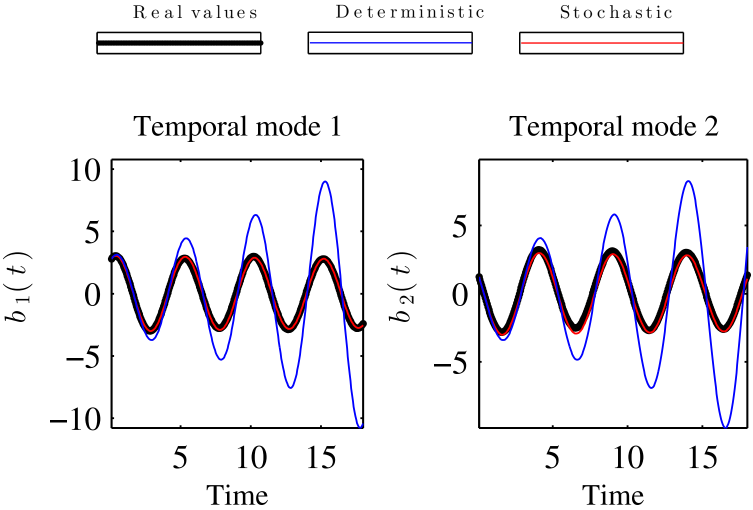


FIGURE : n=2 and Re=3900 (3D flow).

Reconstructions of temporal modes

Modal increase of a estimation :

$$\frac{\partial w}{\partial t} = I + L(w) + C(w,w) + D(a,w) \text{ becomes :}$$

$$\frac{db_i}{dt} \quad = \quad \int_\Omega \phi_i \cdot \left(I + \sum_{p=0}^{n} L(\phi_p) b_p + \sum_{p,q=0}^{n} \big(C(\phi_p, \phi_q) + \alpha_{iq} \, D(\phi_p, z_q) \big) \, b_p b_q \right)$$

RECONSTRUCTIONS OF TEMPORAL MODES

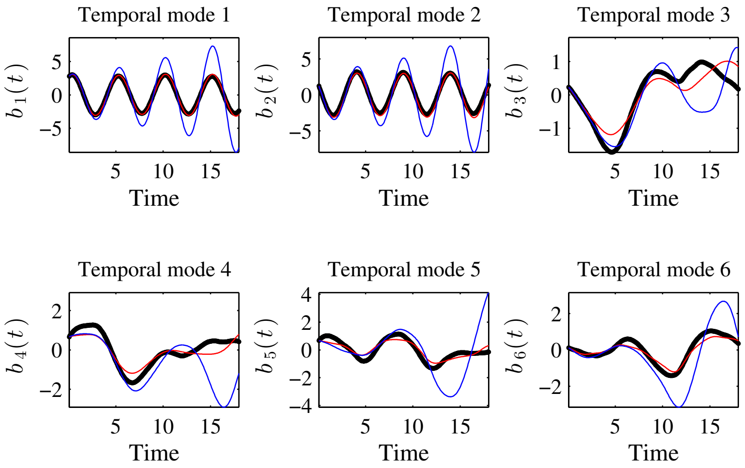


FIGURE : n=6 and Re=3900 (3D flow).

Plan

Summary

- Building of a complete physical fluid dynamic model with few assumptions
- Local variance-covariance matrix $a(x, t)$ projected on large-scale velocity POD-modes :
  Simple autonomous reduced model and physical analysis of turbulence
- Direct application on POD-Galerkin model :
  Encouraging results
  But, the time-decorrelated part of the small-scale random velocity seems not enough energetic
- Corrective coefficient or matrix :
  Simple method with great results

Further work

- w semi-martingale :
  which implies non-Gaussian and partially temporally-correlated random small-scale velocity
- Stochastic oceanic models : Geostrophic, Shallow water, Boussinesq, QG, SQG, ...
- Estimation of a by Kriging estimation of measurements errors (SSH)

QUESTIONS

Thank you for your attention

References

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